Theorem. Given n integers with integer average, some permutation of them is a valid siteswap.

Lemma. Given n numbers which can be rearranged into a valid siteswap, if we change two of the numbers such that the average is still an integer, then the new set can also be rearranged into a valid siteswap.

Proof of Lemma. All arithmetic below is mod n. Assume that the starting sequence is already in valid siteswap order. Let t_i be the i^{th} throw and let $l_i = i + t_i$ be its landing time. We have

$$\begin{array}{cccc} 1 & 2 & \dots & n \\ t_1 & t_2 & \dots & t_n \\ l_1 & l_2 & \dots & l_n \end{array}$$

Let us replace throws t_i and t_j by throws x_i and x_j, such that the resulting sequence still has integer average. Therefore, $t_i + t_j = x_i + x_j$, and so $(i + x_i) + (j + x_j) = l_i + l_j$. $(\star)$

Using $(\star)$, we get: if $i + x_i = l_i$, we already have a siteswap; if $i + x_i = l_j$, we swap $l_i \leftrightarrow l_j$; if $i + x_j = l_i$, we swap $x_i \leftrightarrow x_j$; and if $i + x_j = l_j$, we swap both $x_i \leftrightarrow x_j$ and $l_i \leftrightarrow l_j$.

In any of those cases, we are done. But if none of those hold, let $k = l_i - x_i$. Then $k \neq i$ and $k \neq j$, and k is the time at which throw x_i must happen in order to land at time l_i. We must therefore move the throw that is already occurring at time k. Rearrange the entries in the table:

$$\begin{array}{ccccccc} \dots & i & \dots & j & \dots & k & \dots \\ \dots & x_i & \dots & x_j & \dots & x_k & \dots \\ \dots & l_i & \dots & l_j & \dots & l_k & \dots \end{array} \quad \longrightarrow \quad \begin{array}{ccccccc} \dots & i & \dots & j & \dots & k & \dots \\ \dots & x_k & \dots & x_j & \dots & x_i & \dots \\ \dots & l_j & \dots & l_k & \dots & l_i & \dots \end{array}$$

Column k is valid, so we try to resolve the problems that still exist in columns i and j. Since $k = l_i - x_i = l_k - x_k$, equation $(\star)$ implies $(i + x_k) + (j + x_j) = l_j + l_k$, and thus an equivalent equation holds for the new columns i and j. Therefore, we may relabel the x and l terms and repeat the whole procedure, using the new entries in these columns. We must show that this terminates in a finite number of steps, and it is sufficient to show that each k we find is distinct.

Suppose that we encounter a repeat, and let k be the number with the earliest repeat. Let us continue from the rearrangement we made above, assuming that we haven't fallen into one of the earlier finishing cases. We let $k' = l_j - x_k$ and move l_j into column k'. Since $k = l_k - x_k$ and $l_j \neq l_k$, we know $k' \neq k$. Furthermore, l_j will stay in column k' until the next occurrence of k'. However, since we are assuming that k is the first repeat, l_j must still be in column k' at the time of k's repeat. When we next encounter k, we change the table:

$$\begin{array}{ccccccc} \dots & i & \dots & j & \dots & k & \dots \\ \dots & x'_i & \dots & x_j & \dots & x_i & \dots \\ \dots & l'_i & \dots & l'_j & \dots & l_i & \dots \end{array} \quad \longrightarrow \quad \begin{array}{ccccccc} \dots & i & \dots & j & \dots & k & \dots \\ \dots & x_i & \dots & x_j & \dots & x'_i & \dots \\ \dots & l'_j & \dots & l_i & \dots & l'_i & \dots \end{array}$$

Equation $(\star)$ for the new position here is $(i + x_i) + (j + x_j) = l'_j + l_i$. However, from the original equation $(\star)$, we know that $(i + x_i) + (j + x_j) = l_i + l_j$. Therefore, $l_j = l'_j$. This is a contradiction, since we know that at this point l'_j is in column k'.

Proof of Theorem. Suppose we have the numbers $a_1, \dots, a_n$. Start with the valid siteswap consisting of n 0s. Change the first two 0s to a_1 and $n - a_1$. By the Lemma, some permutation of these is a valid siteswap. Now change that $n - a_1$ to a_2, and the third 0 to $n - a_1 - a_2$, and so on. Since the a_i have integer average, after we change the second-to-last 0 to a_{n-1}, we must also have changed the final 0 to a_n, up to a multiple of n, and this takes a trivial final change.

Acknowledgements. The above is an adaptation of the proof found in *The Mathematics of Juggling* by Burkard Polster.